

# A Numerical Treatment of Generalized Fractional Order Partial Differential Equations and its Coupled Systems: Approximate Solutions, and Operational Matrices Approach

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**Abstract.** A numerical scheme based on the operational matrices of derivatives and integrals of fractional order and two-parametric multi-variables orthogonal shifted Jacobi polynomials is developed to study the approximate solutions for a multi-terms fractional order partial differential equations. The idea is also extended for the generalized class of fractional order coupled systems having terms of mixed partial derivatives type. The Riemann-Liouville fractional integral and Caputo fractional derivative along with their known properties are used to evaluate the fractional integrals and derivatives of multi-variables Jacobi polynomials. With the aid of the operational matrices, the considered fractional order problem and its coupled system are reduced to an algebraic system of equations which are simple in handling using any computational software. Finally, the solution of the algebraic system of equations leads to finding the solution of the considered problem. Validity of the method is established by comparing our Matlab software simulations based obtained results with the exact solutions in the literature, yielding negligible errors. Furthermore, comparative results presented in the literature are extended and improved in this study.

**Keywords:** Multi-variables orthogonal Jacobi polynomials, Operational matrices, Multi-terms coupled systems, Riemann-Liouville integral, Caputo derivative.

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## 1 Introduction

In the literature, it is Niels Henrik Abel who is credited with the very first application of fractional calculus as early as 1823, in the solution of an integral equation arising in the tautochrone

problem formulation [4]. Currently, applications of fractional calculus can be observed in almost every field of science based including, but not limited to, fluid dynamics, physics, chemistry, aerodynamics, signal and image processing, chemical engineering, economics and even psychology, [5–7, 14–17, 22–24, 41, 42]. Based on a preliminary mathematical modeling, determination of exact and/or approximate solutions of fractional order differential equations has proved to be a major emerging research area that has captured the interest of the researchers round the world [19–21, 25–34]. For instance, many scientists have devoted themselves to establishing efficient and reliable numerical schemes for solving fractional order partial differential equations (FOPDEs) using orthogonal polynomials which reduce them into an easily tractable system of algebraic equations.

Certain types of FOPDES where the non-integer order is left arbitrary, have been solved numerically applying such ideas and types of orthogonal polynomials, [8, 9, 18, 35–37]. Recently, particular types of FOPDES have been solved by constructing new operational matrices of fractional derivatives and integrals based on such schemes using both orthogonal and non-orthogonal polynomials, [18, 33, 34, 38–40]. In [8, 9], the numerical schemes are developed with the help of operational matrices based on orthogonal Legendre polynomials.

Motivated by the previously mentioned research works, we opted to develop an efficient numerical scheme able to treat a generalized class of coupled systems of FOPDEs, by developing the operational matrices for a particular kind of polynomials, namely, the Jacobi polynomials, (JPs). Unlike in the case of the Legendre polynomials, the JPs have the ability to approximate the solutions with the aid of two parameters, which is indeed a more generalized way of approximation, since in comparison to the works, [8, 9], herein, the operational matrices developed can deal with the FOPDEs having mixed partial derivatives of fractional order. Moreover, in [18], the operational matrices has been developed using one dimensional JPs. In our case, operational matrices are exhibited for two dimensional JPs. Consequently, among other outcomes of our work, the reader will be pleased to note that the results presented in [8, 9, 18] are not only significantly extended, but also much improved.

In this paper we consider the following generalized FOPDEs

$$\frac{\partial^{\sigma_1} U(x, y)}{\partial x^{\sigma_1}} = a_1 \frac{\partial^{\gamma_1} U(x, y)}{\partial y^{\gamma_1}} + a_2 \frac{\partial^{\varrho_1} U(x, y)}{\partial x^{\varrho_1/2} \partial y^{\varrho_1/2}} + F_1(x, y), \quad (1)$$

subject to the initial conditions

$$U^{(i)}(0, y) = H_i(y), \quad i = 0, 1, \dots, n, \quad (2)$$

and a coupled system of a generalized class of FOPDEs

$$\begin{aligned} \frac{\partial^{\sigma_1} U(x, y)}{\partial x^{\sigma_1}} &= a_1 \frac{\partial^{\gamma_1} U(x, y)}{\partial y^{\gamma_1}} + a_2 \frac{\partial^{\gamma_2} V(x, y)}{\partial x^{\gamma_2}} + a_3 \frac{\partial^{\gamma_3} V(x, y)}{\partial y^{\gamma_3}} + \\ &\quad a_4 \frac{\partial^{\varrho_4} U(x, y)}{\partial x^{\varrho_4/2} \partial y^{\varrho_4/2}} + a_5 \frac{\partial^{\varrho_3} V(x, y)}{\partial x^{\varrho_3/2} \partial y^{\varrho_3/2}} + F_1(x, y), \\ \frac{\partial^{\sigma_2} V(x, y)}{\partial x^{\sigma_2}} &= b_1 \frac{\partial^{\rho_1} V(x, y)}{\partial y^{\rho_1}} + b_2 \frac{\partial^{\rho_2} U(x, y)}{\partial x^{\rho_2}} + b_3 \frac{\partial^{\rho_3} U(x, y)}{\partial y^{\rho_3}} + \\ &\quad b_4 \frac{\partial^{\varrho_1} U(x, y)}{\partial x^{\varrho_1/2} \partial y^{\varrho_1/2}} + b_5 \frac{\partial^{\varrho_2} V(x, y)}{\partial x^{\varrho_2/2} \partial y^{\varrho_2/2}} + F_2(x, y), \end{aligned} \quad (3)$$

subject to the initial conditions

$$U^{(i)}(0, y) = H_i(y), \quad V^{(i)}(0, y) = G_i(y), \quad i = 0, 1, \dots, n, \quad (4)$$

where  $n < \sigma_1, \sigma_2 \leq n+1$ ,  $a_j, b_j$ ,  $j = 1, \dots, 5$  are all real constants and  $U = U(x, y)$ ,  $V = V(x, y)$  are the unknown solutions to be determined. Moreover  $U(x, y), V(x, y), F_1(x, y), F_2(x, y) \in C([0, \eta] \times [0, \eta])$ .

The structure of our paper is as follows: In Section 2, some necessary definitions and mathematical preliminaries from fractional calculus are presented. Moreover, in Section 2, the two parametric orthogonal JPs and applications to functions approximation is presented. In Section 3, operational matrices of fractional order derivatives and integrals based on two parametric orthogonal JPs are developed. In Section 4, an efficient numerical scheme is developed for (1)-(4). In Section 5, four illustrative examples are discussed to show the validity and applicability of the proposed method.

## 2 Preliminaries

For reader's convenience, and a comprehensive treatment, in this section we summarize basic concepts, definitions and results from fractional calculus.

**Definition 2.1.** [10, 11] Given an interval  $[a, b] \subset \mathbb{R}$ , the Riemann-Liouville fractional order integral of order  $\beta \in \mathbb{R}_+$  of a function  $\psi \in (L^1[a, b], \mathbb{R})$  is defined by

$$\mathcal{I}_{a+}^{\beta} \psi(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} \psi(s) ds,$$

provided that the integral on right hand side exists.

**Definition 2.2.** For a given function  $\psi(x) \in C^n[a, b]$ , the Caputo fractional order derivative of order  $\beta > 0$  is defined as

$$D^{\beta} \psi(x) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_a^x \frac{\psi^{(n)}(t)}{(x-t)^{\beta+1-n}} dt, & n-1 < \beta < n, \quad n \in \mathbb{N}, \\ \frac{d^n}{dx^n} \psi(x), & \beta \in \mathbb{N}. \end{cases} \quad (5)$$

For the Caputo derivative we have [1]. We give some properties of fractional derivative from the available resources in [2, 3, 10]:

$D^{\beta} C = 0$ ,  $C$  is a constant,

$$D^{\beta} x^j = \begin{cases} 0, & \text{for } j \in \mathbb{N} \cup \{0\}, \text{ and } j < [\beta], \\ \frac{\Gamma(j+1)}{\Gamma(j+1-\beta)} x^{j-\beta}, & \text{for } j \in \mathbb{N} \cup \{0\}, \text{ and } j \geq [\beta], \text{ or } j \notin \mathbb{N}, \text{ and } j > [\beta]. \end{cases} \quad (6)$$

We use the ceiling function  $[\beta]$  to denote the smallest integer greater than or equal to  $\beta$ , and the floor function  $\lfloor \beta \rfloor$  to denote the largest integer less than or equal to  $\beta$ .

### 2.1 The shifted Jacobi polynomials

The well known two parametric JPs defined on  $[0, \eta]$ , with parameters  $\alpha$  and  $\beta$  are defined by the following relation (see [18])

$$P_{\eta,i}^{(\alpha,\beta)}(x) = \sum_{k=0}^i \frac{(-1)^{i-k} \Gamma(i+\beta+1) \Gamma(i+k+\alpha+\beta+1)}{\Gamma(k+\beta+1) \Gamma(i+\alpha+\beta+1) (i-k)! k! \eta^k} x^k, \quad i = 1, 2, 3, \dots \quad (7)$$

The orthogonality expression is

$$\int_0^1 P_{\eta,i}^{(\alpha,\beta)}(x) P_{\eta,j}^{(\alpha,\beta)}(x) W_{\eta}^{(\alpha,\beta)}(x) dx = R_{\eta,j}^{(\alpha,\beta)} \delta_{i,j}, \quad (8)$$

where

$$W_{\eta}^{(\alpha,\beta)}(x) = (\eta - x)^{\alpha} x^{\beta}, \quad (9)$$

is a weight function. And

$$R_{\eta,j}^{(\alpha,\beta)} \delta_{i,j} = \frac{\eta^{\alpha+\beta+1} \Gamma(j+\alpha+1) \Gamma(j+\beta+1)}{(2j+\alpha+\beta+1) \Gamma(j+1) \Gamma(j+\alpha+\beta+1)}. \quad (10)$$

Which implies that any function  $v(x)$  integrable in  $[0, \eta]$  can be approximated by shifted JPs as follows

$$v(x) \approx \sum_{a=0}^m C_a P_{\eta,a}^{(\alpha,\beta)}(x), \quad (11)$$

as  $m \rightarrow \infty$ , the approximation becomes equal to the exact function. The coefficient  $C_a$  can be easily determined using (8), (9) and (10). The vector expression of (11) is as following

$$v(x) \approx H_M^T \hat{\Psi}_M(x), \quad (12)$$

where  $M = m + 1$ ,  $H_M^T$  is the coefficient vector and  $\hat{\Psi}_M(x)$  is  $M$  terms vector function. The two dimension Jacobi polynomials of order  $M$  on the region  $[0, \eta] \times [0, \eta]$  as a product function of two JPs can be written as

$$P_{\eta,n}^{(\alpha,\beta)}(x, y) = (P_{\eta,i}^{(\alpha,\beta)}(x))(P_{\eta,j}^{(\alpha,\beta)}(y)), \quad n = Mi + j + 1, \quad i = 0, 1, 2, \dots, m, \quad j = 0, 1, 2, \dots, m. \quad (13)$$

The orthogonality expression of  $P_{\eta,n}^{(\alpha,\beta)}(x, y)$  is as following

$$\int_0^{\eta} \int_0^{\eta} (P_{\eta,a}^{(\alpha,\beta)}(x))(P_{\eta,b}^{(\alpha,\beta)}(y))(P_{\eta,c}^{(\alpha,\beta)}(x))(P_{\eta,d}^{(\alpha,\beta)}(y)) W_{\eta}^{(\alpha,\beta)}(x) W_{\eta}^{(\alpha,\beta)}(y) dx dy = R_{\eta,c}^{(\alpha,\beta)} \delta_{a,c} R_{\eta,d}^{(\alpha,\beta)} \delta_{b,d}. \quad (14)$$

Any function square integrable in  $[0, \eta] \times [0, \eta]$  can be approximated by  $M$  terms of the JPs  $P_{\eta,n}^{(\alpha,\beta)}(x, y)$  as follows

$$f(x, y) \approx \sum_{a=0}^m \sum_{b=0}^m C_{ab} (P_{\eta,a}^{(\alpha,\beta)}(x))(P_{\eta,b}^{(\alpha,\beta)}(y)), \quad (15)$$

where  $C_{ab}$  can be obtained by the relation

$$C_{ab} = \frac{1}{R_{\eta,a}^{(\alpha,\beta)} R_{\eta,b}^{(\alpha,\beta)}} \int_0^{\eta} \int_0^{\eta} f(x, y) (P_{\eta,a}^{(\alpha,\beta)}(x))(P_{\eta,b}^{(\alpha,\beta)}(y)) W_{\eta}^{(\alpha,\beta)}(x, y) dx dy, \quad (16)$$

where

$$W_{\eta}^{(\alpha,\beta)}(x, y) = W_{\eta}^{(\alpha,\beta)}(x) W_{\eta}^{(\alpha,\beta)}(y). \quad (17)$$

For simplicity use the notation  $C_n = C_{ab}$ , then (15) can be expressed as

$$f(x, y) \approx \sum_{n=1}^{M^2} C_n P_{\eta,n}^{(\alpha,\beta)}(x, y) = K_{M^2}^T \hat{\Psi}_{M^2}(x, y), \quad \text{with } n = Ma + b + 1 \quad (18)$$

in vector notation, where  $K_{M^2}$  is  $M^2 \times 1$  coefficient column vector and  $\Psi_{M^2}(x, y)$  is  $M^2 \times 1$  column vector of functions defined by

$$\hat{\Psi}_{M^2}(x, y) = \begin{pmatrix} \psi_{11}(x, y) & \cdots & \psi_{1M}(x, y) & \psi_{21}(x, y) & \cdots & \psi_{2M}(x, y) & \cdots & \psi_{MM}(x, y) \end{pmatrix}^T, \quad (19)$$

where  $\psi_{i+1,j+1}(x, y) = (P_{\eta,i}^{(\alpha,\beta)}(x))(P_{\eta,j}^{(\alpha,\beta)}(y))$ ,  $i, j = 0, 1, 2, \dots, m$ .

## 2.2 Error Analysis

In this section we are interested in developing an analytical expression to determine the error of approximation for a sufficiently smooth function  $f(x, y) \in [0, \eta] \times [0, \eta]$ . So, consider a polynomial  $Q_{M \times M}(x, y)$  of degree  $\leq M$  and  $\prod_{M,M}(x, y)$  is a space of Jacobian polynomials. Moreover  $f(x, y)$  is the best approximation in  $\prod_{M,M}(x, y)$ . Then according to the definition of best approximation, we have

$$\|f(x, y) - f_{(M,M)}(x, y)\|_2 \leq \|f(x, y) - Q_{(M,M)}(x, y)\|_2. \quad (20)$$

The inequality in (20) is also satisfied if  $Q_{(M,M)}(x, y)$  is an interpolating polynomial at point  $(x_i, y_j)$ , then by the similar arguments as in [13], the error of the approximation is given by

$$\|f(x, y) - Q_{(M,M)}(x, y)\|_2 \leq (\lambda_1 + \lambda_2 + \lambda_3 \frac{1}{M^{M+1}}) \frac{1}{M^{M+1}},$$

where

$$\lambda_1 = \frac{1}{4} \max_{(x,y) \in [0,1] \times [0,1]} \left| \frac{\partial^{M+1}}{\partial x^{M+1}} f(x, y) \right|, \quad \lambda_2 = \frac{1}{4} \max_{(x,y) \in [0,1] \times [0,1]} \left| \frac{\partial^{M+1}}{\partial y^{M+1}} f(x, y) \right|,$$

$$\lambda_3 = \frac{1}{16} \max_{(x,y) \in [0,1] \times [0,1]} \left| \frac{\partial^{2M+2}}{\partial x^{M+1} \partial y^{M+1}} f(x, y) \right|.$$

We refer the reader to [12] for the proof of the above result.

The results of the next section are very important to develop the numerical scheme. Our main purpose in this section is the development of the new operational matrices of integrals and derivatives using two parametric orthogonal JPs. Using these operational matrices the fractional order problems (1)-(4) are reduced into a problems of solving a system of algebraic equations of Sylvester type which are simple in handling and can be solved by any computational software.

## 3 Operational matrices of Integrations and Differentiations

**Lemma 3.0.1.** *Let  $\hat{\Psi}_{M^2}(x, y)$  be as defined in (19), then the fractional integration of order  $v$  of  $\Psi_{M^2}(x, y)$  w.r.t  $x$  is generalized as*

$$I_x^v(\hat{\Psi}_{M^2}(x, y)) \simeq \hat{P}_{M^2 \times M^2}^{v,x} \hat{\Psi}_{M^2}(x, y), \quad (21)$$

where  $\hat{P}_{M^2 \times M^2}^{v,x}$  is the operational matrix of integration of order  $v$  and is defined as

$$\hat{P}_{M^2 \times M^2}^{v,x} = \begin{pmatrix} \mathcal{U}_{1,1,k} & \mathcal{U}_{1,2,k} & \cdots & \mathcal{U}_{1,\hat{r},k} & \cdots & \mathcal{U}_{1,M^2,k} \\ \mathcal{U}_{2,1,k} & \mathcal{U}_{2,2,k} & \cdots & \mathcal{U}_{2,\hat{r},k} & \cdots & \mathcal{U}_{2,M^2,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{U}_{\hat{q},1,k} & \mathcal{U}_{\hat{q},2,k} & \cdots & \Omega_{\hat{q},\hat{r},k} & \cdots & \mathcal{U}_{\hat{q},M^2,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{U}_{M^2,1,k} & \mathcal{U}_{M^2,2,k} & \cdots & \mathcal{U}_{M^2,\hat{r},k} & \cdots & \mathcal{U}_{M^2,M^2,k} \end{pmatrix},$$

and  $\hat{r} = Mi + j + 1$ ,  $\hat{q} = Ma + b + 1$ ,  $\mathcal{U}_{\hat{q},\hat{r},k} = \sqcup_{i,j,a,b,k}$  for  $i, j, a, b = 0, 1, 2, \dots, m$ ,

$$\sqcup_{i,j,a,b,k} = \sum_{k=0}^a \biguplus_{a,k,v} E_{i,j,b},$$

$$E_{ijb} = \delta_{j,b} \sum_{l=0}^i \frac{(-1)^{i-l} (2i + \alpha + \beta + 1) \Gamma(i+1) \Gamma(i+l + \alpha + \beta + 1) \Gamma(k+v+l + \beta + 1) \Gamma(\alpha+1) \eta^v}{\Gamma(i + \alpha + 1) \Gamma(l + \beta + 1) (i-l)! l! \Gamma(k+v+l + \beta + \alpha + 2)}.$$

And

$$\biguplus_{a,k,v} = \frac{(-1)^{a-k} \Gamma(a + \beta + 1) \Gamma(a + k + \alpha + \beta + 1) \Gamma(1+k)}{\Gamma(k + \beta + 1) \Gamma(a + \alpha + \beta + 1) (a-k)! k! \Gamma(1+k+v) \eta^k}. \quad (22)$$

*Proof.* In order to prove the result, take the fractional integral of order  $v$  of  $P_{\eta,n}^{(\alpha,\beta)}(x,y)$  as defined by (13), with respect to  $x$ , the following relation is obtained

$$I_x^v P_{\eta,n}^{(\alpha,\beta)}(x,y) = I_x^v P_{\eta,a}^{(\alpha,\beta)}(x) P_{\eta,b}^{(\alpha,\beta)}(y) = \sum_{k=0}^a \frac{(-1)^{a-k} \Gamma(a + \beta + 1) \Gamma(a + k + \alpha + \beta + 1)}{\Gamma(k + \beta + 1) \Gamma(a + \alpha + \beta + 1) (a-k)! k! \eta^k} I_x^v x^k P_{\eta,b}^{(\alpha,\beta)}(y),$$

which in view of the definition of fractional integrals takes the form

$$I_x^v P_{\eta,a}^{(\alpha,\beta)}(x) P_{\eta,b}^{(\alpha,\beta)}(y) = \sum_{k=0}^a \frac{(-1)^{a-k} \Gamma(a + \beta + 1) \Gamma(a + k + \alpha + \beta + 1) \Gamma(1+k)}{\Gamma(k + \beta + 1) \Gamma(a + \alpha + \beta + 1) (a-k)! k! \Gamma(1+k+v) \eta^k} x^{k+v} P_{\eta,b}^{(\alpha,\beta)}(y). \quad (23)$$

Approximating  $x^{k+v} P_{\eta,b}^{(\alpha,\beta)}(y)$  by JPs in two variables, it yields

$$x^{k+v} P_{\eta,b}^{(\alpha,\beta)}(y) \approx \sum_{i=0}^m \sum_{j=0}^m E_{i,j,b} P_{\eta,i}^{(\alpha,\beta)}(x) P_{\eta,j}^{(\alpha,\beta)}(y), \quad (24)$$

where  $E_{i,j,b} = \frac{1}{R_{\eta,i}^{(\alpha,\beta)} R_{\eta,j}^{(\alpha,\beta)}} \int_0^\eta \int_0^\eta x^{k+v} P_{\eta,b}^{(\alpha,\beta)}(y) P_{\eta,i}^{(\alpha,\beta)}(x) P_{\eta,j}^{(\alpha,\beta)}(y) W_\eta^{(\alpha,\beta)}(x,y) dx dy$ . And in the light of orthogonality condition, it may be written as

$$E_{i,j,b} = \frac{\delta_{j,b}}{R_{\eta,i}^{(\alpha,\beta)}} \int_0^\eta x^{k+v} P_{\eta,i}^{(\alpha,\beta)}(x) W_\eta^{(\alpha,\beta)}(x) dx. \quad (25)$$

With the help of (7) and (9), (25) can also be written as

$$E_{i,j,b} = \frac{\delta_{j,b}}{R_{\eta,i}^{(\alpha,\beta)}} \sum_{l=0}^i \frac{(-1)^{i-l} \Gamma(i + \beta + 1) \Gamma(i + l + \alpha + \beta + 1)}{\Gamma(l + \beta + 1) \Gamma(i + \alpha + \beta + 1) (i-l)! l! \eta^l} \int_0^\eta x^{k+v+l+\beta} (\eta-x)^\alpha. \quad (26)$$

The integrand in (26) can be determined by the convolution theorem of Laplace transformation. That is

$$\mathcal{L}\left(\int_0^\eta x^{k+v+l+\beta}(\eta-x)^\alpha\right) = \frac{\Gamma(k+v+l+\beta+1)\Gamma(\alpha+1)}{s^{(k+v+l+\beta+\alpha+2)}}. \quad (27)$$

Taking inverse Laplace we get

$$\int_0^\eta x^{k+v+l+\beta}(\eta-x)^\alpha = \frac{\Gamma(k+v+l+\beta+1)\Gamma(\alpha+1)\eta^{(k+v+l+\beta+\alpha+1)}}{\Gamma(k+v+l+\beta+\alpha+1)}. \quad (28)$$

The coefficient in(24) can be written in a generalized form as

$$E_{ijb} = \frac{\delta_{j,b}}{R_{\eta,i}^{(\alpha,\beta)}} \sum_{l=0}^i \frac{(-1)^{i-l}\Gamma(i+\beta+1)\Gamma(i+l+\alpha+\beta+1)\Gamma(k+v+l+\beta+1)\Gamma(\alpha+1)\eta^{(k+v+l+\beta+\alpha+1)}}{\Gamma(l+\beta+1)\Gamma(i+\alpha+\beta+1)(i-l)!l!\eta^l\Gamma(k+v+l+\beta+\alpha+2)}. \quad (29)$$

Using(10) in (29) the most generalized form of the coefficient is as following

$$E_{ijb} = \delta_{j,b} \sum_{l=0}^i \frac{(-1)^{i-l}(2i+\alpha+\beta+1)\Gamma(i+1)\Gamma(i+l+\alpha+\beta+1)\Gamma(k+v+l+\beta+1)\Gamma(\alpha+1)\eta^v}{\Gamma(i+\alpha+1)\Gamma(l+\beta+1)(i-l)!l!\Gamma(k+v+l+\beta+\alpha+2)}. \quad (30)$$

For convenience in handling the notations let

$$\biguplus_{a,k,v} = \frac{(-1)^{a-k}\Gamma(a+\beta+1)\Gamma(a+k+\alpha+\beta+1)\Gamma(1+k)}{\Gamma(k+\beta+1)\Gamma(a+\alpha+\beta+1)(a-k)!k!\Gamma(1+k+v)\eta^k}. \quad (31)$$

In the light of (24), (30) and (62), (23) can be written as

$$I_x^v P_{\eta,a}^{(\alpha,\beta)}(x) P_{\eta,b}^{(\alpha,\beta)}(y) = \sum_{k=0}^a \biguplus_{a,k,v} \sum_{i=0}^m \sum_{j=0}^m E_{i,j,b} P_{\eta,i}^{(\alpha,\beta)}(x) P_{\eta,j}^{(\alpha,\beta)}(y). \quad (32)$$

Or

$$I_x^v P_{\eta,a}^{(\alpha,\beta)}(x) P_{\eta,b}^{(\alpha,\beta)}(y) = \sum_{i=0}^m \sum_{j=0}^m \sum_{k=0}^a \biguplus_{a,k,v} E_{i,j,b} P_{\eta,i}^{(\alpha,\beta)}(x) P_{\eta,j}^{(\alpha,\beta)}(y). \quad (33)$$

Let

$$\bigsqcup_{i,j,a,b,k} = \sum_{k=0}^a \biguplus_{a,k,v} E_{i,j,b}, \quad (34)$$

we get

$$I_x^v P_{\eta,a}^{(\alpha,\beta)}(x) P_{\eta,b}^{(\alpha,\beta)}(y) = \sum_{i=0}^m \sum_{j=0}^m \bigsqcup_{i,j,a,b,k} P_{\eta,i}^{(\alpha,\beta)}(x) P_{\eta,j}^{(\alpha,\beta)}(y). \quad (35)$$

Using the notations  $\hat{r} = Mi + j + 1$ ,  $\hat{q} = Ma + b + 1$ ,  $\mathcal{U}_{\hat{q},\hat{r},k} = \bigsqcup_{i,j,b,a,k}$  for  $i, j, a, b = 0, 1, 2, 3, \dots, m$ , we get the desired result.  $\square$

**Lemma 3.0.2.** *let  $\hat{\Psi}_{M^2}(x, y)$  be as defined in (19), then the derivative of order  $\sigma$  of  $\hat{\Psi}_{M^2}(x, y)$  w.r.t  $x$  is given by*

$$D_x^\sigma(\hat{\Psi}_{M^2}(x, y)) \simeq H_{M^2 \times M^2}^{\sigma,x} \hat{\Psi}_{M^2}(x, y), \quad (36)$$

where  $H_{M^2 \times M^2}^{\sigma, y}$  is the operational matrix of derivative of order  $\sigma$  and is defined as

$$H_{M^2 \times M^2}^{\sigma, x} = \begin{pmatrix} \mathcal{U}_{1,1,k} & \mathcal{U}_{1,2,k} & \cdots & \mathcal{U}_{1,\hat{r},k} & \cdots & \mathcal{U}_{1,M^2,k} \\ \mathcal{U}_{2,1,k} & \mathcal{U}_{2,2,k} & \cdots & \mathcal{U}_{2,\hat{r},k} & \cdots & \mathcal{U}_{2,M^2,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{U}_{\hat{q},1,k} & \mathcal{U}_{\hat{q},2,k} & \cdots & \mathcal{U}_{\hat{q},\hat{r},k} & \cdots & \mathcal{U}_{\hat{q},M^2,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{U}_{M^2,1,k} & \mathcal{U}_{M^2,2,k} & \cdots & \mathcal{U}_{M^2,\hat{r},k} & \cdots & \mathcal{U}_{M^2,M^2,k} \end{pmatrix}, \quad (37)$$

and  $\hat{q} = Mi + j + 1$ ,  $\hat{r} = Ma + b + 1$ ,  $\mathcal{U}_{\hat{q},\hat{r},k} = \sqcup_{i,j,a,b,k}$  for  $i, j, a, b = 0, 1, 2, \dots, m$ ,

$$\sqcup_{i,j,a,b,k} = \sum_{k=\lceil \sigma \rceil}^a \biguplus_{a,k,\sigma} E_{i,j,b}, \quad (38)$$

$$E_{ijb} = \delta_{j,b} \sum_{l=0}^i \frac{(-1)^{i-l} (2i + \alpha + \beta + 1) \Gamma(i+1) \Gamma(i+l+\alpha+\beta+1) \Gamma(k-\sigma+l+\beta+1) \Gamma(\alpha+1) \eta^\sigma}{\Gamma(i+\alpha+1) \Gamma(l+\beta+1) (i-l)! l! \Gamma(k-\sigma+l+\beta+\alpha+2)}. \quad (39)$$

And

$$\biguplus_{a,k,\sigma} = \frac{(-1)^{a-k} \Gamma(a+\beta+1) \Gamma(a+k+\alpha+\beta+1) \Gamma(1+k)}{\Gamma(k+\beta+1) \Gamma(a+\alpha+\beta+1) (a-k)! k! \Gamma(1+k-\sigma) \eta^k}. \quad (40)$$

*Proof.* In order to prove the result, take the fractional derivative of order  $\sigma$  of  $P_{\eta,n}^{(\alpha,\beta)}(x, y)$  as defined by (13) with respect to  $x$ , the following relation is obtained

$$D_x^\sigma P_{\eta,n}^{(\alpha,\beta)}(x, y) = D_x^\sigma P_{\eta,a}^{(\alpha,\beta)}(x) P_{\eta,b}^{(\alpha,\beta)}(y) = \sum_{k=0}^a \frac{(-1)^{a-k} \Gamma(a+\beta+1) \Gamma(a+k+\alpha+\beta+1)}{\Gamma(k+\beta+1) \Gamma(a+\alpha+\beta+1) (a-k)! k! \eta^k} D_x^\sigma x^k P_{\eta,b}^{(\alpha,\beta)}(y),$$

using Definition 2.2 it takes the form

$$D_x^\sigma P_{\eta,a}^{(\alpha,\beta)}(x) P_{\eta,b}^{(\alpha,\beta)}(y) = \sum_{k=\lceil v \rceil}^a \frac{(-1)^{a-k} \Gamma(a+\beta+1) \Gamma(a+k+\alpha+\beta+1) \Gamma(1+k)}{\Gamma(k+\beta+1) \Gamma(a+\alpha+\beta+1) (a-k)! k! \Gamma(1+k-\sigma) \eta^k} x^{k-\sigma} P_{\eta,b}^{(\alpha,\beta)}(y). \quad (41)$$

Approximating  $x^{k-\sigma} P_{\eta,b}^{(\alpha,\beta)}(y)$  by JPs in two variables, we obtain

$$x^{k-\sigma} P_{\eta,b}^{(\alpha,\beta)}(y) \approx \sum_{i=0}^m \sum_{j=0}^m E_{i,j,b} P_{\eta,i}^{(\alpha,\beta)}(x) P_{\eta,j}^{(\alpha,\beta)}(y), \quad (42)$$

where  $E_{i,j,b} = \frac{1}{R_{\eta,i}^{(\alpha,\beta)} R_{\eta,j}^{(\alpha,\beta)}} \int_0^\eta \int_0^\eta x^{k-\sigma} P_{\eta,b}^{(\alpha,\beta)}(y) P_{\eta,i}^{(\alpha,\beta)}(x) P_{\eta,j}^{(\alpha,\beta)}(y) W_\eta^{(\alpha,\beta)}(x, y) dx dy$ . And in the light of orthogonality condition, it may be written as

$$E_{i,j,b} = \frac{\delta_{j,b}}{R_{\eta,i}^{(\alpha,\beta)}} \int_0^\eta x^{k-\sigma} P_{\eta,i}^{(\alpha,\beta)}(x) W_\eta^{(\alpha,\beta)}(x) dx. \quad (43)$$

With the help of (7) and (9), (43) can be written as

$$E_{i,j,b} = \frac{\delta_{j,b}}{R_{\eta,i}^{(\alpha,\beta)}} \sum_{l=0}^i \frac{(-1)^{i-l} \Gamma(i+\beta+1) \Gamma(i+l+\alpha+\beta+1)}{\Gamma(l+\beta+1) \Gamma(i+\alpha+\beta+1) (i-l)! l! \eta^l} \int_0^\eta x^{k-\sigma+l+\beta} (\eta-x)^\alpha. \quad (44)$$



The integrand in (44) can be determined by convolution theorem of Laplace transformation. That is

$$\mathcal{L}\left(\int_0^\eta x^{k-\sigma+l+\beta}(\eta-x)^\alpha\right) = \frac{\Gamma(k-\sigma+l+\beta+1)\Gamma(\alpha+1)}{s^{(k-\sigma+l+\beta+\alpha+2)}}. \quad (45)$$

Taking inverse Laplace we get

$$\int_0^\eta x^{k-\sigma+l+\beta}(\eta-x)^\alpha = \frac{\Gamma(k-\sigma+l+\beta+1)\Gamma(\alpha+1)\eta^{(k-\sigma+l+\beta+\alpha+1)}}{\Gamma(k-\sigma+l+\beta+\alpha+1)}. \quad (46)$$

The coefficient in (42) can be written in a generalized form as

$$E_{ijb} = \delta_{j,b} \sum_{l=0}^i \frac{(-1)^{i-l}(2i+\alpha+\beta+1)\Gamma(i+1)\Gamma(i+l+\alpha+\beta+1)\Gamma(k-\sigma+l+\beta+1)\Gamma(\alpha+1)\eta^{-\sigma}}{\Gamma(i+\alpha+1)\Gamma(l+\beta+1)(i-l)!l!\Gamma(k-\sigma+l+\beta+\alpha+2)}. \quad (47)$$

For convenience in handling the notations, let

$$\biguplus_{a,k,\sigma} = \frac{(-1)^{a-k}\Gamma(a+\beta+1)\Gamma(a+k+\alpha+\beta+1)\Gamma(1+k)}{\Gamma(k+\beta+1)\Gamma(a+\alpha+\beta+1)(a-k)!k!\Gamma(1+k-\sigma)\eta^k}. \quad (48)$$

In the light of (42), (47), (48), (41) takes the form

$$D_x^\sigma P_{\eta,a}^{(\alpha,\beta)}(x)P_{\eta,b}^{(\alpha,\beta)}(y) = \sum_{k=\lceil\sigma\rceil}^a \biguplus_{a,k,\sigma} \sum_{i=0}^m \sum_{j=0}^m E_{i,j,b} P_{\eta,i}^{(\alpha,\beta)}(x)P_{\eta,j}^{(\alpha,\beta)}(y). \quad (49)$$

Or

$$D_x^\sigma P_{\eta,a}^{(\alpha,\beta)}(x)P_{\eta,b}^{(\alpha,\beta)}(y) = \sum_{i=0}^m \sum_{j=0}^m \sum_{k=\lceil\sigma\rceil}^a \biguplus_{a,k,\sigma} E_{i,j,b} P_{\eta,i}^{(\alpha,\beta)}(x)P_{\eta,j}^{(\alpha,\beta)}(y). \quad (50)$$

Let

$$\bigsqcup_{i,j,a,b,k} = \sum_{k=\lceil\sigma\rceil}^a \biguplus_{a,k,\sigma} E_{i,j,b}, \quad (51)$$

we get

$$D_x^\sigma P_{\eta,a}^{(\alpha,\beta)}(x)P_{\eta,b}^{(\alpha,\beta)}(y) = \sum_{i=0}^m \sum_{j=0}^m \bigsqcup_{i,j,a,b,k} P_{\eta,i}^{(\alpha,\beta)}(x)P_{\eta,j}^{(\alpha,\beta)}(y). \quad (52)$$

Using the notations  $\hat{r} = Mi + j + 1$ ,  $\hat{q} = Ma + b + 1$ ,  $\mathcal{U}_{\hat{q},\hat{r},k} = \bigsqcup_{i,j,b,a,k}$  for  $i, j, a, b = 0, 1, 2, 3, \dots, m$ , we get the desired result.  $\square$

**Lemma 3.0.3.** *let  $\hat{\Psi}_{M^2}(x, y)$  be as defined in (19), then the derivative of order  $\sigma$  of  $\hat{\Psi}_{M^2}(x, y)$  w.r.t  $y$  is given by*

$$D_y^\sigma(\hat{\Psi}_{M^2}(x, y)) \simeq H_{M^2 \times M^2}^{\sigma,y} \hat{\Psi}_{M^2}(x, y), \quad (53)$$

where  $H_{M^2 \times M^2}^{\sigma,y}$  is the operational matrix of derivative of order  $\sigma$  w.r.t  $y$  and is defined as

$$H_{M^2 \times M^2}^{\sigma,y} = \begin{pmatrix} \mathcal{U}_{1,1,k} & \mathcal{U}_{1,2,k} & \cdots & \mathcal{U}_{1,\hat{r},k} & \cdots & \mathcal{U}_{1,M^2,k} \\ \mathcal{U}_{2,1,k} & \mathcal{U}_{2,2,k} & \cdots & \mathcal{U}_{2,\hat{r},k} & \cdots & \mathcal{U}_{2,M^2,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{U}_{\hat{q},1,k} & \mathcal{U}_{\hat{q},2,k} & \cdots & \mathcal{U}_{\hat{q},\hat{r},k} & \cdots & \mathcal{U}_{\hat{q},M^2,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{U}_{M^2,1,k} & \mathcal{U}_{M^2,2,k} & \cdots & \mathcal{U}_{M^2,\hat{r},k} & \cdots & \mathcal{U}_{M^2,M^2,k} \end{pmatrix}, \quad (54)$$

and  $\hat{r} = Mi + j + 1$ ,  $\hat{q} = Ma + b + 1$ ,  $\mathcal{U}_{\hat{q},\hat{r},k} = \sqcup_{i,j,a,b,k} \text{ for } i, j, a, b = 0, 1, 2, \dots, m$ ,

$$\sqcup_{i,j,a,b,k} = \sum_{k=\lceil \sigma \rceil}^a \biguplus_{a,k,\sigma} E_{i,j,b}, \quad (55)$$

$$E_{ijb} = \delta_{i,a} \sum_{l=0}^i \frac{(-1)^{i-l} (2i + \alpha + \beta + 1) \Gamma(i+1) \Gamma(i+l + \alpha + \beta + 1) \Gamma(k - \sigma + l + \beta + 1) \Gamma(\alpha + 1) \eta^\sigma}{\Gamma(i + \alpha + 1) \Gamma(l + \beta + 1) (i-l)! l! \Gamma(k - \sigma + l + \beta + \alpha + 2)}, \quad (56)$$

and

$$\biguplus_{a,k,\sigma} = \frac{(-1)^{a-k} \Gamma(a + \beta + 1) \Gamma(a + k + \alpha + \beta + 1) \Gamma(1 + k)}{\Gamma(k + \beta + 1) \Gamma(a + \alpha + \beta + 1) (a-k)! k! \Gamma(1 + k - \sigma) \eta^k}. \quad (57)$$

*Proof.* The proof of this lemma is similar as Lemma 3.0.2.  $\square$

**Lemma 3.0.4.** *let  $\hat{\Psi}_{M^2}(x, y)$  be as defined in (19), then the mixed partial derivative of order  $\sigma$  of  $\hat{\Psi}_{M^2}(x, y)$  w.r.t  $xy$  is given by*

$$\frac{\partial^\sigma}{\partial x^{(\sigma/2)} \partial y^{(\sigma/2)}} (\hat{\Psi}_{M^2}(x, y)) \simeq H_{M^2 \times M^2}^{\sigma, x, y} \hat{\Psi}_{M^2}(x, y), \quad (58)$$

where  $H_{M^2 \times M^2}^{\sigma, x, y}$  is the operational matrix of mixed derivative of order  $\sigma$  w.r.t  $xy$ , and is defined as

$$H_{M^2 \times M^2}^{\sigma, x, y} = \begin{pmatrix} \mathcal{U}_{1,1,k,l} & \mathcal{U}_{1,2,k,l} & \cdots & \mathcal{U}_{1,\hat{r},k,l} & \cdots & \mathcal{U}_{1,M^2,k,l} \\ \mathcal{U}_{2,1,k,l} & \mathcal{U}_{2,2,k,l} & \cdots & \mathcal{U}_{2,\hat{r},k,l} & \cdots & \mathcal{U}_{2,M^2,k,l} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{U}_{\hat{q},1,k,l} & \mathcal{U}_{\hat{q},2,k,l} & \cdots & \mathcal{U}_{\hat{q},\hat{r},k,l} & \cdots & \mathcal{U}_{\hat{q},M^2,k,l} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{U}_{M^2,1,k,l} & \mathcal{U}_{M^2,2,k,l} & \cdots & \mathcal{U}_{M^2,\hat{r},k,l} & \cdots & \mathcal{U}_{M^2,M^2,k,l} \end{pmatrix}, \quad (59)$$

and  $\hat{r} = Mi + j + 1$ ,  $\hat{q} = Ma + b + 1$ ,  $\mathcal{U}_{\hat{q},\hat{r},k,l} = \sqcup_{i,j,a,b,k,l,\eta} \text{ for } i, j, a, b = 0, 1, 2, \dots, m$ ,

$$\sqcup_{i,j,a,b,k,l,\eta} = \sum_{k=\lceil \frac{\sigma}{2} \rceil}^a \sum_{l=\lceil \frac{\sigma}{2} \rceil}^b \widehat{\Upsilon}_{(a,k,\alpha,\beta,\eta)} \widehat{\Upsilon}_{(b,l,\alpha,\beta,\eta)} E_{(i,j,k,l)}, \quad (60)$$

$$E_{(i,j,k,l)} = \frac{1}{R_{\eta,i}^{(\alpha,\beta)} R_{\eta,j}^{(\alpha,\beta)}} \sum_{l'=0}^i \sum_{k'=0}^j \Upsilon_{(i,l',\alpha,\beta,\eta)} \Upsilon_{(j,k',\alpha,\beta,\eta)} A_{(l',k,\alpha,\beta,\eta)} B_{(k',l,\alpha,\beta,\eta)}, \quad (61)$$

$$\Upsilon_{(i,l',\alpha,\beta,\eta)} = \frac{(-1)^{i-l'} \Gamma(i + \beta + 1) \Gamma(i + l' + \alpha + \beta + 1)}{\Gamma(l' + \beta + 1) \Gamma(i + \alpha + \beta + 1) (a - l')! l'! \eta^{l'}}, \quad (62)$$

$$A_{(l',k,\alpha,\beta,\eta)} = \frac{\Gamma(l' + \beta + k - \sigma/2 + 1) \Gamma(\alpha + 1)}{\Gamma(l' + \beta + k - \sigma/2 + \alpha + 1)} \eta^{l' + \beta + k - \sigma/2 + \alpha + 1}, \quad (63)$$

and

$$\widehat{\Upsilon}_{(a,k,\alpha,\beta,\eta)} = \frac{(-1)^{a-k} \Gamma(a + \beta + 1) \Gamma(a + k + \alpha + \beta + 1)}{\Gamma(k + \beta + 1) \Gamma(a + \alpha + \beta + 1) (a - k)! \eta^k \Gamma(1 + k - \sigma/2)}. \quad (64)$$

*Proof.* In order to prove the result take the fractional derivative of order  $\sigma$  of  $P_{\eta,n}^{(\alpha,\beta)}(x,y)$  as defined by (13) with respect to  $xy$ , the following relation is obtained

$$\begin{aligned} \frac{\partial^\sigma}{\partial x^{(\sigma/2)} \partial y^{(\sigma/2)}} P_{\eta,n}^{(\alpha,\beta)}(x,y) &= \frac{\partial^\sigma}{\partial x^{(\sigma/2)} \partial y^{(\sigma/2)}} P_{\eta,a}^{(\alpha,\beta)}(x) P_{\eta,b}^{(\alpha,\beta)}(y), \\ &= \sum_{k=0}^a \Upsilon_{(a,k,\alpha,\beta,\eta)} \sum_{l=0}^b \Upsilon_{(b,l,\alpha,\beta,\eta)} D_x^{\sigma/2} x^k D_y^{\sigma/2} y^l, \end{aligned}$$

where

$$\begin{aligned} \Upsilon_{(a,k,\alpha,\beta,\eta)} &= \frac{(-1)^{a-k} \Gamma(a+\beta+1) \Gamma(a+k+\alpha+\beta+1)}{\Gamma(k+\beta+1) \Gamma(a+\alpha+\beta+1) (a-k)! k! \eta^k}, \\ \frac{\partial^\sigma}{\partial x^{(\sigma/2)} \partial y^{(\sigma/2)}} P_{\eta,a}^{(\alpha,\beta)}(x) P_{\eta,b}^{(\alpha,\beta)}(y) &= \sum_{k=\lceil \frac{\sigma}{2} \rceil}^a \Upsilon_{(a,k,\alpha,\beta,\eta)} \sum_{l=\lceil \frac{\sigma}{2} \rceil}^b \Upsilon_{(b,l,\alpha,\beta,\eta)} \times \\ &\quad \frac{\Gamma(1+k) \Gamma(1+l)}{\Gamma(1+k-\sigma/2) \Gamma(1+l-\sigma/2)} x^{(k-\sigma/2)} y^{(l-\sigma/2)}, \end{aligned} \quad (65)$$

approximating  $x^{(k-\sigma/2)} y^{(l-\sigma/2)}$  with two dimensional JPs, we have

$$x^{(k-\sigma/2)} y^{(l-\sigma/2)} = \sum_{i=0}^m \sum_{j=0}^m E_{(i,j,k,l)} P_{\eta,i}^{(\alpha,\beta)}(x) P_{\eta,j}^{(\alpha,\beta)}(y), \quad (66)$$

where

$$E_{(i,j,k,l)} = \frac{1}{R_{\eta,i}^{(\alpha,\beta)} R_{\eta,j}^{(\alpha,\beta)}} \int_0^\eta \int_0^\eta x^{(k-\sigma/2)} y^{(l-\sigma/2)} P_{\eta,i}^{(\alpha,\beta)}(x) P_{\eta,j}^{(\alpha,\beta)}(y) W_\eta^{(\alpha,\beta)}(x,y) dx dy. \quad (67)$$

Which after expansion and further simplification takes the form

$$S_{(i,j,k,l)} = \frac{1}{R_{\eta,i}^{(\alpha,\beta)} R_{\eta,j}^{(\alpha,\beta)}} \sum_{l'=0}^i \sum_{k'=0}^j \Upsilon_{(i,l',\alpha,\beta,\eta)} \Upsilon_{(j,k',\alpha,\beta,\eta)} \int_0^\eta y^{(k'+\beta+l-\sigma/2)} (\eta-y)^\alpha dy \int_0^\eta x^{(l'+\beta+k-\sigma/2)} (\eta-x)^\alpha dx. \quad (68)$$

The integrands in the above expression can be easily evaluated using convolution theorem of Laplace transformation, that is

$$\mathcal{L} \left( \int_0^\eta x^{(l'+\beta+k-\sigma/2)} (\eta-x)^\alpha dx \right) = \frac{\Gamma(l'+\beta+k-\sigma/2+1) \Gamma(\alpha+1)}{s^{(l'+\beta+k-\sigma/2+\alpha+2)}}.$$

Taking inverse Laplace, we get

$$\int_0^\eta x^{(l'+\beta+k-\sigma/2)} (\eta-x)^\alpha dx = \frac{\Gamma(l'+\beta+k-\sigma/2+1) \Gamma(\alpha+1) \eta^{(l'+\beta+k-\sigma/2+\alpha+1)}}{\Gamma(l'+\beta+k-\sigma/2+\alpha+1)}.$$

Similarly

$$\int_0^\eta y^{(k'+\beta+l-\sigma/2)} (\eta-y)^\alpha dy = \frac{\Gamma(k'+\beta+l-\sigma/2+1) \Gamma(\alpha+1) \eta^{(k'+\beta+l-\sigma/2+\alpha+1)}}{\Gamma(k'+\beta+l-\sigma/2+\alpha+1)}.$$

The generalized coefficient of (66) can be written as

$$E_{(i,j,k,l)} = \frac{1}{R_{\eta,i}^{(\alpha,\beta)} R_{\eta,j}^{(\alpha,\beta)}} \sum_{l'=0}^i \sum_{k'=0}^j \Upsilon_{(i,l',\alpha,\beta,\eta)} \Upsilon_{(j,k',\alpha,\beta,\eta)} A_{(l',k,\alpha,\beta,\eta)} B_{(k',l,\alpha,\beta,\eta)}, \quad (69)$$

where

$$A_{(l',k,\alpha,\beta,\eta)} = \frac{\Gamma(l' + \beta + k - \sigma/2 + 1) \Gamma(\alpha + 1)}{\Gamma(l' + \beta + k - \sigma/2 + \alpha + 1)} \eta^{l' + \beta + k - \sigma/2 + \alpha + 1}, \quad (70)$$

and

$$B_{(k',l,\alpha,\beta,\eta)} = \frac{\Gamma(k' + \beta + l - \sigma/2 + 1) \Gamma(\alpha + 1)}{\Gamma(k' + \beta + l - \sigma/2 + \alpha + 1)} \eta^{k' + \beta + l - \sigma/2 + \alpha + 1}. \quad (71)$$

Now using (69) along with (66), (70) and (71) in (65), we get

$$\frac{\partial^\sigma P_{\eta,a}^{(\alpha,\beta)}(x) P_{\eta,b}^{(\alpha,\beta)}(y)}{\partial x^{(\sigma/2)} \partial y^{(\sigma/2)}} = \sum_{i=0}^m \sum_{j=0}^m \sum_{k=\lceil \frac{\sigma}{2} \rceil}^a \sum_{l=\lceil \frac{\sigma}{2} \rceil}^b \widehat{\Upsilon}_{(a,k,\alpha,\beta,\eta)} \widehat{\Upsilon}_{(b,l,\alpha,\beta,\eta)} E_{(i,j,k,l)} P_{\eta,i}^{(\alpha,\beta)}(x) P_{\eta,j}^{(\alpha,\beta)}(y), \quad (72)$$

where

$$\widehat{\Upsilon}_{(a,k,\alpha,\beta,\eta)} = \frac{(-1)^{a-k} \Gamma(a + \beta + 1) \Gamma(a + k + \alpha + \beta + 1)}{\Gamma(k + \beta + 1) \Gamma(a + \alpha + \beta + 1) (a - k)! \eta^k \Gamma(1 + k - \sigma/2)}, \quad (73)$$

and

$$\widehat{\Upsilon}_{(b,l,\alpha,\beta,\eta)} = \frac{(-1)^{b-l} \Gamma(b + \beta + 1) \Gamma(b + l + \alpha + \beta + 1)}{\Gamma(l + \beta + 1) \Gamma(b + \alpha + \beta + 1) (b - l)! \eta^l \Gamma(1 + l - \sigma/2)}. \quad (74)$$

Let

$$\bigsqcup_{i,j,a,b,k,l,\eta} = \sum_{k=\lceil \frac{\sigma}{2} \rceil}^a \sum_{l=\lceil \frac{\sigma}{2} \rceil}^b \widehat{\Upsilon}_{(a,k,\alpha,\beta,\eta)} \widehat{\Upsilon}_{(b,l,\alpha,\beta,\eta)} E_{(i,j,k,l)}, \quad (75)$$

we get

$$\frac{\partial^\sigma}{\partial x^{(\sigma/2)} \partial y^{(\sigma/2)}} P_{\eta,a}^{(\alpha,\beta)}(x) P_{\eta,b}^{(\alpha,\beta)}(y) = \sum_{i=0}^m \sum_{j=0}^m \bigsqcup_{i,j,a,b,k,l,\eta} P_{\eta,i}^{(\alpha,\beta)}(x) P_{\eta,j}^{(\alpha,\beta)}(y). \quad (76)$$

Using the notations  $\hat{r} = Mi + j + 1$ ,  $\hat{q} = Ma + b + 1$ ,  $\mathcal{U}_{\hat{q},\hat{r},k,l} = \bigsqcup_{i,j,b,a,k,l,\eta}$  for  $i, j, a, b = 0, 1, 2, 3, \dots, m$ , we get the desired result.  $\square$

## 4 Applications of the operational matrices of integration and derivative to FOPDEs

In this section we are interested to generalize a new technique to approximate the solution of a generalized class of fractional order partial differential equations.

### 4.1 Solution of FOPDEs by JPs

First we show the solution method for fractional order partial differential equation of the form

$$\frac{\partial^{\sigma_1} U(x, y)}{\partial x^{\sigma_1}} = a_1 \frac{\partial^{\gamma_1} U(x, y)}{\partial y^{\gamma_1}} + a_2 \frac{\partial^{\varrho_1} U(x, y)}{\partial x^{\varrho_1/2} \partial y^{\varrho_1/2}} + F_1(x, y), \quad (77)$$

subject to the initial conditions

$$U^{(i)}(0, y) = H_i(y), \quad i = 0, 1 \dots n. \quad (78)$$

We seek the solution of the problem in terms of shifted JPs such that the following holds

$$\frac{\partial^{\sigma_1} U(x, y)}{\partial x^{\sigma_1}} = \mathbf{K}_{M^2} \Psi_{M^2}(x, y). \quad (79)$$

By application of fractional integral of order  $\sigma$  and making use of operational matrix of integration we can write

$$U(x, y) - \sum_{i=0}^n c_i x^i = \mathbf{K}_{M^2} \mathbf{P}_{M^2 \times M^2}^{(\sigma, x)} \Psi_{M^2}(x, y). \quad (80)$$

Using the initial condition we can write

$$U(x, y) = \mathbf{K}_{M^2} \mathbf{P}_{M^2 \times M^2}^{(\sigma, x)} \Psi_{M^2}(x, y) + \mathbf{F}_{M^2}^1 \Psi_{M^2}(x, y). \quad (81)$$

Now using the operational matrix of differentiation we can write

$$\frac{\partial^{\gamma_1} U(x, y)}{\partial y^{\gamma_1}} = \mathbf{K}_{M^2} \mathbf{P}_{M^2 \times M^2}^{(\sigma, x)} \mathbf{D}_{M^2 \times M^2}^{(\gamma_1, y)} \Psi_{M^2}(x, y) + \mathbf{F}_{M^2}^1 \mathbf{D}_{M^2 \times M^2}^{(\gamma_1, y)} \Psi_{M^2}(x, y), \quad (82)$$

$$\frac{\partial^{\varrho_1} U(x, y)}{\partial x^{\varrho_1/2} \partial y^{\varrho_1/2}} = \mathbf{K}_{M^2} \mathbf{P}_{M^2 \times M^2}^{(\sigma, x)} \mathbf{D}_{M^2 \times M^2}^{(\varrho_1, x, y)} \Psi_{M^2}(x, y) + \mathbf{F}_{M^2}^1 \mathbf{D}_{M^2 \times M^2}^{(\varrho_1, x, y)} \Psi_{M^2}(x, y). \quad (83)$$

Now using the estimates (83), (82) and (80) in (77) we get the following system of algebraic equations

$$\mathbf{K}_{M^2} = \mathbf{K}_{M^2} \mathbf{P}_{M^2 \times M^2}^{(\sigma, x)} (a_1 \mathbf{D}_{M^2 \times M^2}^{(\gamma_1, y)} + a_2 \mathbf{D}_{M^2 \times M^2}^{(\varrho_1, x, y)}) + \mathbf{F}_{M^2}^1 (a_1 \mathbf{D}_{M^2 \times M^2}^{(\gamma_1, y)} + a_2 \mathbf{D}_{M^2 \times M^2}^{(\varrho_1, x, y)}) + \mathbf{F}_{M^2}^2, \quad (84)$$

which is a generalized system of algebraic equations and can be easily solved for the unknown matrix.

## 4.2 Solution of Coupled system of FOPDEs by JPs

Consider the following class of fractional order partial differential equations

$$\begin{aligned} \frac{\partial^{\sigma_1} U(x, y)}{\partial x^{\sigma_1}} &= a_1 \frac{\partial^{\gamma_1} U(x, y)}{\partial y^{\gamma_1}} + a_2 \frac{\partial^{\gamma_2} V(x, y)}{\partial x^{\gamma_2}} + a_3 \frac{\partial^{\gamma_3} V(x, y)}{\partial y^{\gamma_3}} + \\ &\quad a_4 \frac{\partial^{\varrho_4} U(x, y)}{\partial x^{\varrho_4/2} \partial y^{\varrho_4/2}} + a_5 \frac{\partial^{\varrho_3} V(x, y)}{\partial x^{\varrho_3/4} \partial y^{\varrho_3/2}} + F_1(x, y), \\ \frac{\partial^{\sigma_2} V(x, y)}{\partial x^{\sigma_2}} &= b_1 \frac{\partial^{\rho_1} V(x, y)}{\partial y^{\rho_1}} + b_2 \frac{\partial^{\rho_2} U(x, y)}{\partial x^{\rho_2}} + b_3 \frac{\partial^{\rho_3} U(x, y)}{\partial y^{\rho_3}} + \\ &\quad b_4 \frac{\partial^{\varrho_1} U(x, y)}{\partial x^{\varrho_1/2} \partial y^{\varrho_1/2}} + b_5 \frac{\partial^{\varrho_2} V(x, y)}{\partial x^{\varrho_2/2} \partial y^{\varrho_2/2}} + F_2(x, y), \end{aligned} \quad (85)$$

subject to the initial conditions

$$U^{(i)}(0, y) = h_i(y), \quad V^{(i)}(0, y) = g_i(y), \quad i = 0, 1 \dots n, \quad (86)$$

where  $n < \sigma_1, \sigma_2 \leq n+1$ ,  $a_i, b_i$ ,  $i = 1, 2, \dots, 5$  are all real constants and  $U = U(x, y)$ ,  $V = V(x, y)$  are the unknown solutions of the system to be determined. Moreover  $U(x, y), V(x, y) \in C([0, \eta] \times [0, \eta])$ , and  $F_1(x, y), F_2(x, y) \in C([0, \eta] \times [0, \eta])$ . Assume that the solution of the problem is in the form of shifted Jacobi series, such that the following holds

$$\frac{\partial^{\sigma_1} U(x, y)}{\partial x^{\sigma_1}} = \mathbf{K}_{M^2} \Psi_{M^2}(x, y), \quad \frac{\partial^{\sigma_2} V(x, y)}{\partial x^{\sigma_2}} = \mathbf{L}_{M^2} \Psi_{M^2}(x, y). \quad (87)$$

Now applying integral of order  $\sigma_1$  and  $\sigma_2$  on the corresponding equations we get

$$\begin{aligned} U(x, y) - \sum_{i=0}^n c_i x^i &= \mathbf{K}_{M^2} \mathbf{P}_{M^2 \times M^2}^{(\sigma_1, x)} \Psi_{M^2}(x, y), \\ V(x, y) - \sum_{i=0}^n d_i x^i &= \mathbf{L}_{M^2} \mathbf{P}_{M^2 \times M^2}^{(\sigma_2, x)} \Psi_{M^2}(x, y). \end{aligned} \quad (88)$$

Using the initial conditions we can get the constants of integration and a generalized relation for the solution can be obtained as

$$\begin{aligned} U(x, y) &= \mathbf{K}_{M^2} \mathbf{P}_{M^2 \times M^2}^{(\sigma_1, x)} \Psi_{M^2}(x, y) + \mathbf{F}_{M^2}^1 \Psi_{M^2}(x, y), \\ V(x, y) &= \mathbf{L}_{M^2} \mathbf{P}_{M^2 \times M^2}^{(\sigma_2, x)} \Psi_{M^2}(x, y) + \mathbf{F}_{M^2}^2 \Psi_{M^2}(x, y), \end{aligned} \quad (89)$$

where  $\mathbf{F}_{M^2}^1 \Psi_{M^2}(x, y) = \sum_{i=0}^n h_i(y) x^i$ ,  $\mathbf{F}_{M^2}^2 \Psi_{M^2}(x, y) = \sum_{i=0}^n g_i(y) x^i$ . For simplicity of notation we can write

$$\mathbf{K}_{M^2} \mathbf{P}_{M^2 \times M^2}^{(\sigma_1, x)} + \mathbf{F}_{M^2}^1 = \widehat{\mathbf{K}}_{M^2}, \quad \mathbf{L}_{M^2} \mathbf{P}_{M^2 \times M^2}^{(\sigma_2, x)} + \mathbf{F}_{M^2}^2 = \widehat{\mathbf{L}}_{M^2}. \quad (90)$$

And hence we can write it as

$$U(x, y) = \widehat{\mathbf{K}}_{M^2} \Psi_{M^2}(x, y), \quad V(x, y) = \widehat{\mathbf{L}}_{M^2} \Psi_{M^2}(x, y). \quad (91)$$

Using relation (91) and operational matrices of derivatives we can get the following estimates

$$\begin{aligned} \frac{\partial^{\gamma_1} U(x, y)}{\partial y^{\gamma_1}} &= \widehat{\mathbf{K}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\gamma_1, y)} \Psi_{M^2}(x, y), & \frac{\partial^{\gamma_2} V(x, y)}{\partial x^{\gamma_2}} &= \widehat{\mathbf{L}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\gamma_2, x)} \Psi_{M^2}(x, y), \\ \frac{\partial^{\gamma_3} V(x, y)}{\partial y^{\gamma_3}} &= \widehat{\mathbf{L}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\gamma_3, y)} \Psi_{M^2}(x, y), & \frac{\partial^{\varrho_4} U(x, y)}{\partial x^{\varrho_4/2} \partial y^{\varrho_4/2}} &= \widehat{\mathbf{K}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\varrho_4, x, y)} \Psi_{M^2}(x, y), \\ \frac{\partial^{\varrho_3} V(x, y)}{\partial x^{\varrho_3/2} \partial y^{\varrho_3/2}} &= \widehat{\mathbf{L}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\varrho_3, x, y)} \Psi_{M^2}(x, y), & \frac{\partial^{\rho_1} V(x, y)}{\partial y^{\rho_1}} &= \widehat{\mathbf{L}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\rho_1, y)} \Psi_{M^2}(x, y), \\ \frac{\partial^{\rho_2} U(x, y)}{\partial x^{\rho_2}} &= \widehat{\mathbf{K}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\rho_2, x)} \Psi_{M^2}(x, y), & \frac{\partial^{\rho_3} U(x, y)}{\partial y^{\rho_3}} &= \widehat{\mathbf{K}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\rho_3, y)} \Psi_{M^2}(x, y), \\ \frac{\partial^{\varrho_1} U(x, y)}{\partial x^{\varrho_1/2} \partial y^{\varrho_1/2}} &= \widehat{\mathbf{K}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\varrho_1, x, y)} \Psi_{M^2}(x, y), & \frac{\partial^{\varrho_2} V(x, y)}{\partial x^{\varrho_2/2} \partial y^{\varrho_2/2}} &= \widehat{\mathbf{L}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\varrho_2, x, y)} \Psi_{M^2}(x, y), \\ F_1(x, y) &= \mathbf{F}_{M^2}^3 \Psi_{M^2}(x, y), & F_2(x, y) &= \mathbf{F}_{M^2}^4 \Psi_{M^2}(x, y). \end{aligned} \quad (92)$$

$$\begin{aligned}
\begin{bmatrix} \mathbf{K}_{M^2} \Psi_{M^2}(x, y) \\ \mathbf{L}_{M^2} \Psi_{M^2}(x, y) \end{bmatrix} &= \begin{bmatrix} a_1 \widehat{\mathbf{K}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\gamma_1, y)} \Psi_{M^2}(x, y) \\ b_1 \widehat{\mathbf{L}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\rho_1, y)} \Psi_{M^2}(x, y) \end{bmatrix} + \begin{bmatrix} a_2 \widehat{\mathbf{L}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\gamma_2, x)} \Psi_{M^2}(x, y) \\ b_2 \widehat{\mathbf{K}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\rho_2, x)} \Psi_{M^2}(x, y) \end{bmatrix} \\
&+ \begin{bmatrix} a_3 \widehat{\mathbf{L}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\gamma_3, y)} \Psi_{M^2}(x, y) \\ b_3 \widehat{\mathbf{K}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\rho_3, y)} \Psi_{M^2}(x, y) \end{bmatrix} + \begin{bmatrix} a_4 \widehat{\mathbf{K}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\varrho_4, x, y)} \Psi_{M^2}(x, y) \\ b_5 \widehat{\mathbf{L}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\varrho_2, x, y)} \Psi_{M^2}(x, y) \end{bmatrix} + \begin{bmatrix} \mathbf{F}_{M^2}^3 \Psi_{M^2}(x, y) \\ \mathbf{F}_{M^2}^4 \Psi_{M^2}(x, y) \end{bmatrix} \\
&+ \begin{bmatrix} a_5 \widehat{\mathbf{L}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\varrho_3, x, y)} \Psi_{M^2}(x, y) \\ b_4 \widehat{\mathbf{K}}_{M^2} \mathbf{D}_{M^2 \times M^2}^{(\varrho_1, x, y)} \Psi_{M^2}(x, y) \end{bmatrix}.
\end{aligned} \tag{93}$$

$$\begin{aligned}
\begin{bmatrix} \mathbf{K}_{M^2} & \mathbf{L}_{M^2} \end{bmatrix} \hat{A} &= \begin{bmatrix} \widehat{\mathbf{K}}_{M^2} & \widehat{\mathbf{L}}_{M^2} \end{bmatrix} \begin{bmatrix} a_1 \mathbf{D}_{M^2 \times M^2}^{(\gamma_1, y)} & O_{M^2 \times M^2} \\ O_{M^2 \times M^2} & b_1 \mathbf{D}_{M^2 \times M^2}^{(\rho_1, y)} \end{bmatrix} \hat{A} \\
&+ \begin{bmatrix} \widehat{\mathbf{K}}_{M^2} & \widehat{\mathbf{L}}_{M^2} \end{bmatrix} \begin{bmatrix} O_{M^2 \times M^2} & b_2 \mathbf{D}_{M^2 \times M^2}^{(\rho_2, x)} \\ a_2 \mathbf{D}_{M^2 \times M^2}^{(\gamma_2, x)} & O_{M^2 \times M^2} \end{bmatrix} \hat{A} \\
&+ \begin{bmatrix} \widehat{\mathbf{K}}_{M^2} & \widehat{\mathbf{L}}_{M^2} \end{bmatrix} \begin{bmatrix} O_{M^2 \times M^2} & b_3 \mathbf{D}_{M^2 \times M^2}^{(\rho_3, y)} \\ a_3 \mathbf{D}_{M^2 \times M^2}^{(\gamma_3, y)} & O_{M^2 \times M^2} \end{bmatrix} \hat{A} \\
&+ \begin{bmatrix} \widehat{\mathbf{K}}_{M^2} & \widehat{\mathbf{L}}_{M^2} \end{bmatrix} \begin{bmatrix} a_4 \mathbf{D}_{M^2 \times M^2}^{(\varrho_4, x, y)} & O_{M^2 \times M^2} \\ O_{M^2 \times M^2} & b_5 \mathbf{D}_{M^2 \times M^2}^{(\varrho_2, x, y)} \end{bmatrix} \hat{A} \\
&\begin{bmatrix} \widehat{\mathbf{K}}_{M^2} & \widehat{\mathbf{L}}_{M^2} \end{bmatrix} \begin{bmatrix} O_{M^2 \times M^2} & b_4 \mathbf{D}_{M^2 \times M^2}^{(\varrho_1, x, y)} \\ a_5 \mathbf{D}_{M^2 \times M^2}^{(\varrho_3, x, y)} & O_{M^2 \times M^2} \end{bmatrix} \hat{A} + \begin{bmatrix} \mathbf{F}_{M^2}^3 & \mathbf{F}_{M^2}^4 \end{bmatrix} \hat{A},
\end{aligned} \tag{94}$$

where

$$\hat{A} = \begin{bmatrix} \Psi_{M^2}(x, y) & O_{M^2} \\ O_{M^2} & \Psi_{M^2}(x, y) \end{bmatrix}.$$

$O_{M^2}$  is a column vector of zeros, and  $O_{M^2 \times M^2}$  is matrix having all entries equal to zero.

Now canceling out the common term and after a short simplification we can write (94) as

$$\begin{bmatrix} \mathbf{K}_{M^2} & \mathbf{L}_{M^2} \end{bmatrix} - \begin{bmatrix} \widehat{\mathbf{K}}_{M^2} & \widehat{\mathbf{L}}_{M^2} \end{bmatrix} \widehat{H} - [\mathbf{F}_{M^2}^3 \mathbf{F}_{M^2}^4] = 0, \tag{95}$$

where

$$\widehat{H} = \begin{bmatrix} a_1 \mathbf{D}_{M^2 \times M^2}^{(\gamma_1, y)} + a_4 \mathbf{D}_{M^2 \times M^2}^{(\varrho_4, x, y)} & b_2 \mathbf{D}_{M^2 \times M^2}^{(\rho_2, x)} + b_3 \mathbf{D}_{M^2 \times M^2}^{(\rho_3, y)} + b_4 \mathbf{D}_{M^2 \times M^2}^{(\varrho_1, x, y)} \\ a_2 \mathbf{D}_{M^2 \times M^2}^{(\gamma_2, x)} + a_3 \mathbf{D}_{M^2 \times M^2}^{(\gamma_3, y)} + a_5 \mathbf{D}_{M^2 \times M^2}^{(\varrho_3, x, y)} & b_5 \mathbf{D}_{M^2 \times M^2}^{(\varrho_2, x, y)} + b_1 \mathbf{D}_{M^2 \times M^2}^{(\rho_1, y)} \end{bmatrix}.$$

Using the values of  $\widehat{\mathbf{K}}_{M^2}$  and  $\widehat{\mathbf{L}}_{M^2}$  we can write the above equations as

$$\begin{bmatrix} \mathbf{K}_{M^2} & \mathbf{L}_{M^2} \end{bmatrix} - \begin{bmatrix} \mathbf{K}_{M^2} & \mathbf{L}_{M^2} \end{bmatrix} \widehat{H}_1 - [\mathbf{F}_{M^2}^1 \mathbf{F}_{M^2}^2] \widehat{H} - [\mathbf{F}_{M^2}^3 \mathbf{F}_{M^2}^4] = 0, \tag{96}$$

$$\text{where } \widehat{H}_1 = \begin{bmatrix} \mathbf{P}_{M^2 \times M^2}^{(\sigma_1, x)} \mathbf{D}_1 & \mathbf{P}_{M^2 \times M^2}^{(\sigma_1, x)} \mathbf{D}_2 \\ \mathbf{P}_{M^2 \times M^2}^{(\sigma_1, x)} \mathbf{D}_3 & \mathbf{P}_{M^2 \times M^2}^{(\sigma_1, x)} \mathbf{D}_4 \end{bmatrix},$$

where  $\mathbf{D}_1 = (a_1 \mathbf{D}_{M^2 \times M^2}^{(\gamma_1, y)} + a_4 \mathbf{D}_{M^2 \times M^2}^{(\varrho_4, x, y)})$ ,  $\mathbf{D}_2 = (b_2 \mathbf{D}_{M^2 \times M^2}^{(\rho_2, x)} + b_3 \mathbf{D}_{M^2 \times M^2}^{(\rho_3, y)} + b_4 \mathbf{D}_{M^2 \times M^2}^{(\varrho_1, x, y)})$ ,  $\mathbf{D}_3 =$

$(a_2 \mathbf{D}_{M^2 \times M^2}^{(\gamma_2, x)} + a_3 \mathbf{D}_{M^2 \times M^2}^{(\gamma_3, y)} + a_5 \mathbf{D}_{M^2 \times M^2}^{(\varrho_3, x, y)})$ , and  $\mathbf{D}_4 = (b_5 \mathbf{D}_{M^2 \times M^2}^{(\varrho_2, x, y)} + b_1 \mathbf{D}_{M^2 \times M^2}^{(\rho_1, y)})$ . Now it can be easily seen that (96) is a system of easily solvable matrix equation which can be easily solved for the unknown  $[\mathbf{K}_{M^2} \mathbf{L}_{M^2}]$ . On using the values of  $[\mathbf{K}_{M^2} \mathbf{L}_{M^2}]$  in (91) will lead us to the approximate solution of the problem.

## 5 Illustrative Examples

We show the applicability of the proposed method with solving some test problems. Note that all the simulations are carried out using 5 Ghz processor. All the results are displayed using plots and tables. The algorithm is designed such that it can be easily simulated with any computational software. We use *MatLab* for calculation and simulations.

**Example 1.** As a first example consider the following fractional order differential equations

$$\frac{\partial^{1.8} U(x, y)}{\partial x^{1.8}} = \frac{\partial U(x, y)}{\partial y} + 2 \frac{\partial^2 U(x, y)}{\partial x \partial y} + F_1(x, y), \quad (97)$$

where  $F_1(x, y) = \frac{39239842694707435}{18014398509481984} x^{\frac{1}{5}} \sin(y) - x^2 \cos(y) - 2x \cos(y)$ . The exact solution of the problem is  $U(x, y) = x^2 \sin(y)$ . We approximate the solution of this problem with the proposed method and observed that the solution is very accurate. The comparison of exact and approximate solution at  $M = 5$  is shown in Fig 1. The absolute difference of exact and approximate solution is shown in Fig 2.

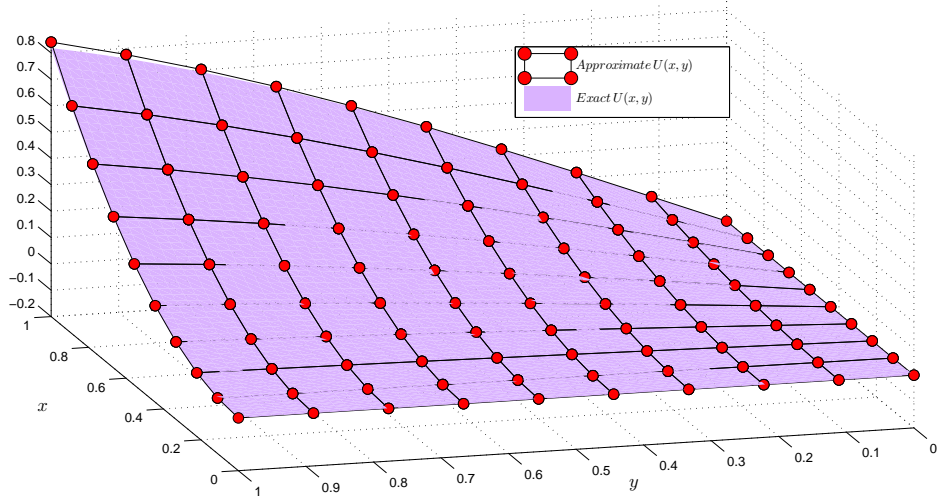


Figure 1: Comparing approximate solution  $U(x, y)$  with the exact solution of the system obtained using  $\alpha = 2$ ,  $\beta = 2$  and  $M = 5$ .



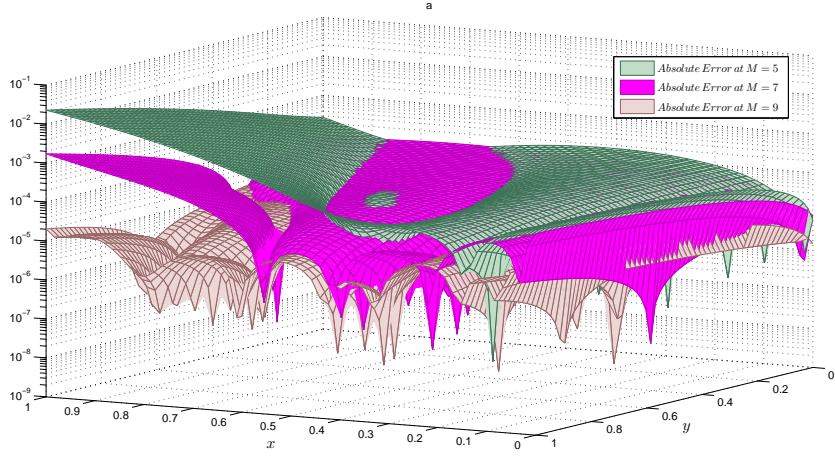


Figure 2: Absolute error of Example1 at different scale levels.

**Example 2.** As a first example consider the following coupled system of integer order partial differential equations.

$$\begin{aligned}\frac{\partial^2 U}{\partial x^2} &= 2\frac{\partial U}{\partial y} + 2\frac{\partial V}{\partial x} + 3\frac{\partial V}{\partial y} + 5\frac{\partial^2 U}{\partial x \partial y} + 6\frac{\partial^2 V}{\partial x \partial y} + f_1(x, y), \\ \frac{\partial^2 V}{\partial x^2} &= 3\frac{\partial V}{\partial y} + 4\frac{\partial U}{\partial x} + 2\frac{\partial U}{\partial y} + 3\frac{\partial^2 U}{\partial x \partial y} - \frac{\partial^2 V}{\partial x \partial y} + f_2(x, y).\end{aligned}\quad (98)$$

Where the source terms  $f_1$  and  $f_2$  are defined as

$$f_1 = -8x^4y^3 - 80x^3y^3 + 15x^3y^2 + 12x^2y^4 + 6x^2y^3 + 99x^2y^2 - 6x^2y - 6xy^3 - 4xy^2 - 24xy, \quad (99)$$

$$f_2 = -8x^4y^3 - 16x^3y^4 - 48x^3y^3 + 15x^3y^2 + 12x^2y^3 + 18x^2y^2 - 6x^2y - 6xy^3 + 4xy + 2y^2. \quad (100)$$

Subject to the initial conditions

$$U(0, y) = U'(0, y) = 0, \quad V(0, y) = V'(0, y) = 0. \quad (101)$$

The exact solution of the above problem is

$$U(x, y) = (xy)^4 - (xy)^3,$$

and

$$V(x, y) = (xy)^2 - (xy)^3.$$

One can easily check it by direct substitution. We approximate the solution of the problem at different scale levels. We compare exact and approximate solution of the problem depicting in Figures (3) and (4) for parameters  $\alpha = 2$ ,  $\beta = 2$  and scale level  $M = 5$ , observing that the exact solution matches well with the approximate solution. We also approximate the absolute error at different scale level  $M = 5, 6$  and at different points of the plane showing in Table (1). Figure (5) showing that with the increase of the scale level the error of approximation (absolute error) decreases significantly.

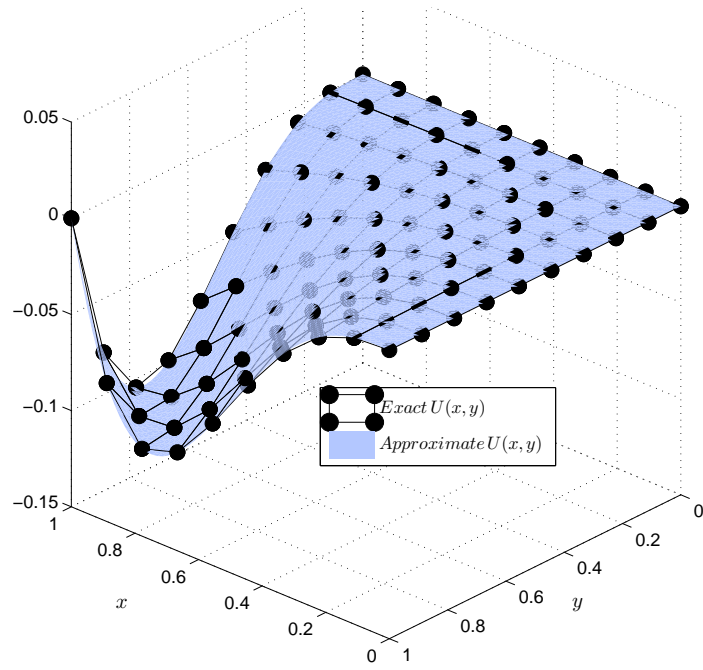


Figure 3: Comparing approximate solution  $U(x, y)$  with the exact solution of the system obtained using  $\alpha = 2$ ,  $\beta = 2$  and  $M = 5$ .

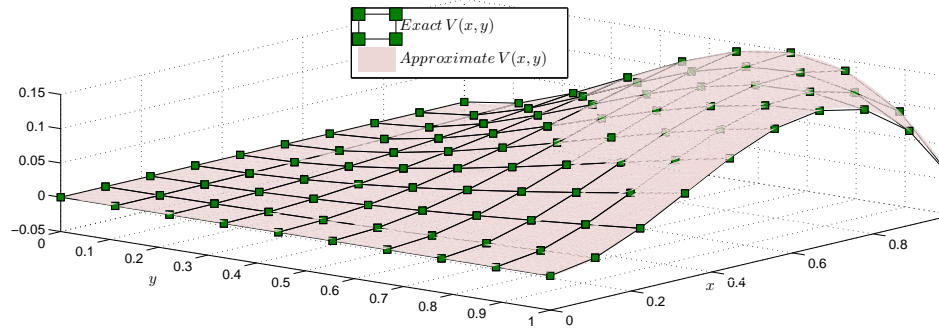


Figure 4: Comparing approximate solution  $V(x, y)$  with the exact solution of the system obtained using  $\alpha = 2$ ,  $\beta = 2$  and  $M = 5$ .

**Table 1:** Absolute difference between  $U(x, y)$  and  $V(x, y)$  at scale value  $M = 5$  and 6 at different points of the plane.

$(x, y)$	$ U - U_5 $	$ U - U_6 $	$ V - V_5 $	$ V - V_6 $
(0.1, 0.1)	$1.707E - 13$	$1.425E - 13$	$1.826E - 13$	$1.678E - 13$
(0.1, 0.5)	$1.956E - 13$	$1.596E - 13$	$2.638E - 13$	$1.839E - 13$
(0.1, 0.9)	$2.236E - 13$	$1.773E - 13$	$2.126E - 13$	$2.019E - 13$
(0.5, 0.1)	$1.873E - 12$	$6.38E - 13$	$4.451E - 12$	$6.368E - 13$
(0.5, 0.5)	$1.88E - 12$	$7.858E - 13$	$3.781E - 12$	$7.52E - 13$
(0.5, 0.9)	$2.092E - 13$	$9.619E - 13$	$3.933E - 12$	$9.051E - 13$
(0.9, 0.1)	$1.27E - 11$	$9.363E - 12$	$1.65E - 11$	$8.394E - 12$
(0.9, 0.5)	$1.307E - 11$	$1.029E - 11$	$1.8E - 11$	$9.996E - 12$
(0.9, 0.9)	$1.306E - 11$	$1.418E - 12$	$1.746E - 11$	$1.209E - 11$

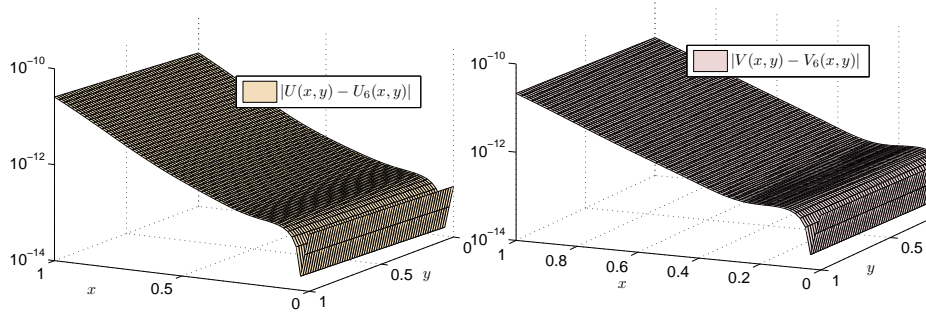


Figure 5: Error of approximation in  $U(x, y)$  and  $V(x, y)$  by keeping  $\alpha = 2$ ,  $\beta = 2$ ,  $M = 5$  and 6.

**Example 3.** As a second example consider the following coupled system of fractional differential equations

$$\begin{aligned}\frac{\partial^{1.8}U}{\partial x^{1.8}} &= \frac{\partial^{0.8}U}{\partial y^{0.8}} + \frac{\partial^2U}{\partial x\partial y} + \frac{\partial^2V}{\partial x\partial y} + f_1(x, y), \\ \frac{\partial^{1.8}V}{\partial x^{1.8}} &= \frac{\partial^{0.8}V}{\partial y^{0.8}} + \frac{\partial^2U}{\partial x\partial y} + \frac{\partial^2V}{\partial x\partial y} + f_2(x, y).\end{aligned}\tag{102}$$

$$\begin{aligned}f_1(x, y) = & 3x^2y^3(2y-2)(x-1)^3 + 3x^3y^3(2y-2)(x-1)^2 + 4x^4y^3(2x-2)(y-1)^3 + \\ & 3x^4y^4(2x-2)(y-1)^2 + 9x^2y^2(x-1)^3(y-1)^2 + 9x^3y^2(x-1)^2(y-1)^2 + \\ & 16x^3y^3(x-1)^2(y-1)^3 + 12x^3y^4(x-1)^2(y-1)^2 - \\ & \frac{89181460669789625x^{\frac{11}{5}}y^4(y-1)^3(125x^2-175x+56)}{504403158265495552} + \\ & \frac{445907303348948125x^4y^{\frac{16}{5}}(x-1)^2(4375y^3-11625y^2+10075y-2821)}{406548945561989414912},\end{aligned}\tag{103}$$

$$\begin{aligned}
f_2(x, y) = & 3x^2y^3(2y-2)(x-1)^3 + 3x^3y^3(2y-2)(x-1)^2 + 12x^4y^3(2x-2)(y-1)^3 + \\
& 9x^4y^4(2x-2)(y-1)^2 + 9x^2y^2(x-1)^3(y-1)^2 + 9x^3y^2(x-1)^2(y-1)^2 + \\
& 36x^3y^4(x-1)^2(y-1)^2 + \frac{89181460669789625x^3y^{\frac{11}{5}}(x-1)^3(125y^2-210y+84)}{3026418949592973312} \\
& - \frac{17836292133957925x^{\frac{6}{5}}y^3(y-1)^2(1250x^3-2625x^2+1680x-308)}{1008806316530991104} \\
& + 48x^3y^3(x-1)^2(y-1)^3.
\end{aligned} \tag{104}$$

The exact solution of this problem is

$$U(x, y) = -x^4y^4(x-1)^2(y-1)^3,$$

and

$$U(x, y) = -x^3y^3(x-1)^3(y-1)^2.$$

One can easily check it by direct substitution. We approximate the solution of the problem with proposed method, and as expected we found that the approximate solution matches very well with the exact solution. We display the exact and approximate solution of the considered problem at parameters  $\alpha = 1$ ,  $\beta = 1$  and scale level  $M = 6$  and  $M = 4$  respectively. The results are displayed at Figures (6) and (7). We observe that the method gives a very high accurate estimate of the solution and the error of approximation (absolute error) decreases significantly by increasing of scale level  $M$ , depicting in Figures (8) and (9).

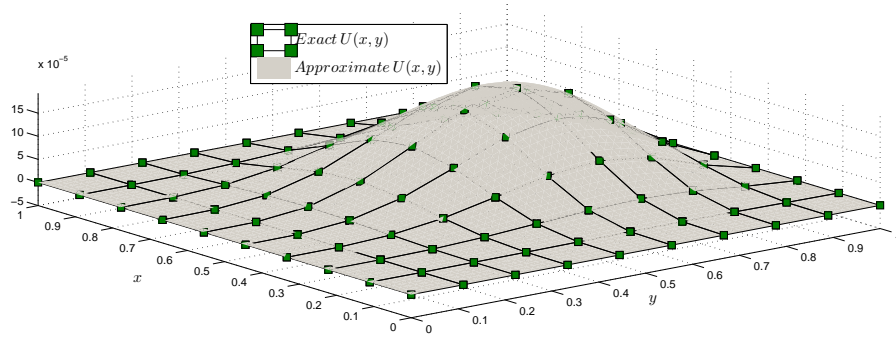


Figure 6: Comparing approximate solution  $U(x, y)$  with the exact solution of the system obtained using  $\alpha = 1, \beta = 1$  and  $M = 6$ .

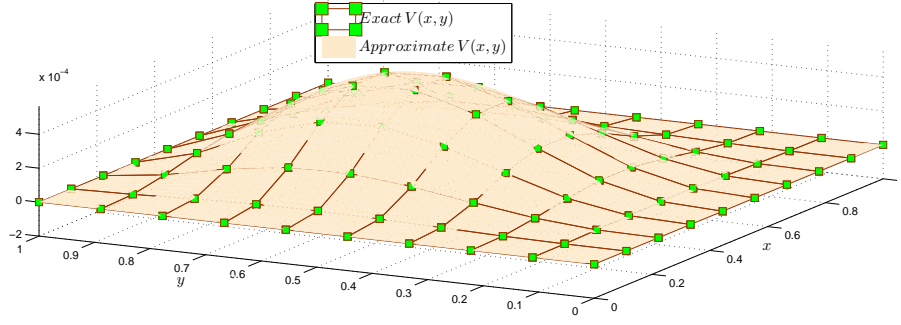


Figure 7: Comparing approximate solution  $V(x, y)$  with the exact solution of the system obtained using  $\alpha = 1, \beta = 1$  and  $M = 4$ .

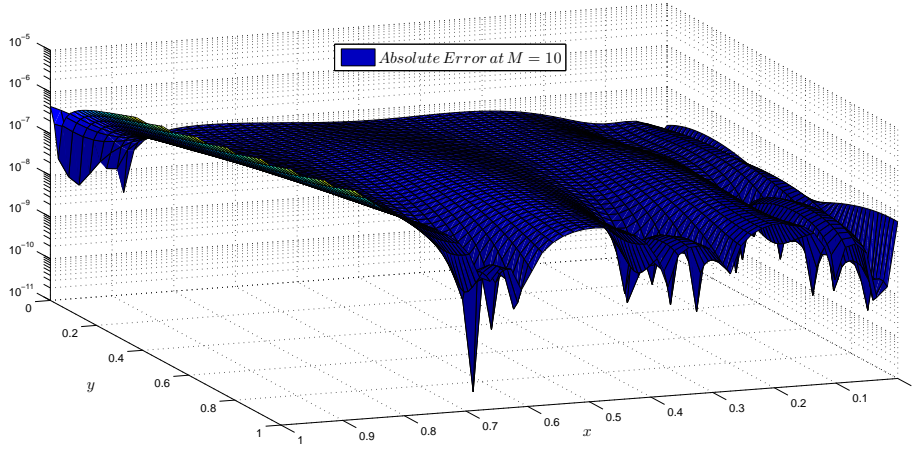


Figure 8: Error of approximation in  $U(x, y)$  by keeping  $\alpha = 1, \beta = 1$  and  $M = 10$ .

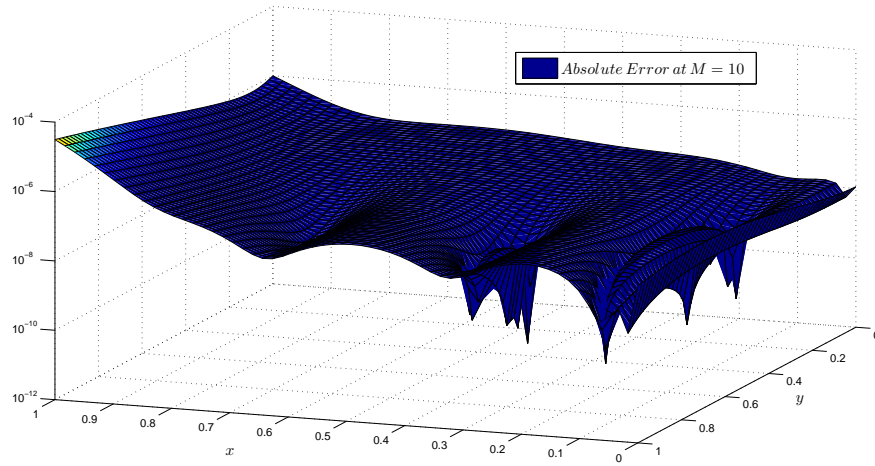


Figure 9: Error of approximation in  $U(x, y)$  by keeping  $\alpha = 1, \beta = 1$  and  $M = 10$ .

**Example 4.**

$$\begin{aligned}\frac{\partial^{1.8}U}{\partial x^{1.8}} &= \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} + \frac{\partial^2 V}{\partial x \partial y} + f_1(x, y), \\ \frac{\partial^2 V}{\partial x^2} &= \frac{\partial^{0.8}U}{\partial x^{0.8}} + \frac{\partial U}{\partial y} + \frac{\partial^2 U}{\partial x \partial y} + f_2(x, y).\end{aligned}\tag{105}$$

$$f_1(x, y) = \frac{39239842694707435 x^{\frac{1}{5}} e^y}{18014398509481984} - y^2 e^x - x^2 e^y - 2 y e^x,\tag{106}$$

$$f_2(x, y) = y^2 e^x - x^2 e^y - 8 x e^y.\tag{107}$$

The exact solution of the problem is

$$U(x, y) = x^2 e^y,$$

and

$$V(x, y) = y^2 e^x.$$

One can easily check it by direct substitution. We approximate the solution of the problem with proposed method, and as expected we found that the approximate solution matches very well with the exact solution. We display the exact and approximate solution of the considered problem at parameters  $\alpha = 4$ ,  $\beta = 3$  and scale level  $M = 8$ . The results are displayed at Figures (10) and (11). We observe that the method gives a very high accurate estimate of the solution and the error of approximation (absolute error) decreases significantly by increasing of scale level  $M$ , depicting in Figures (12) and 13.

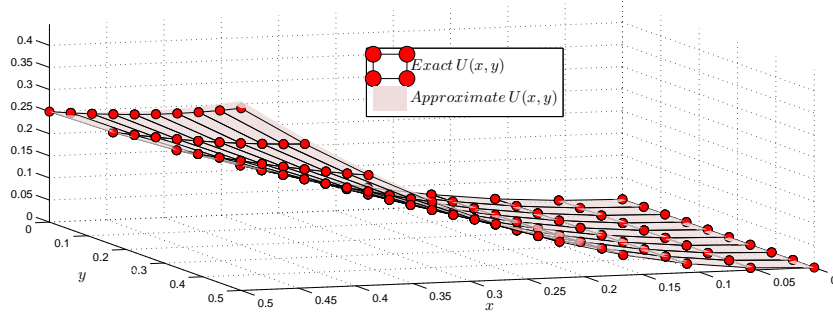


Figure 10: Comparing approximate solution  $U(x, y)$  with the exact solution of the system obtained using  $\alpha = 4, \beta = 3$  and  $M = 8$ .



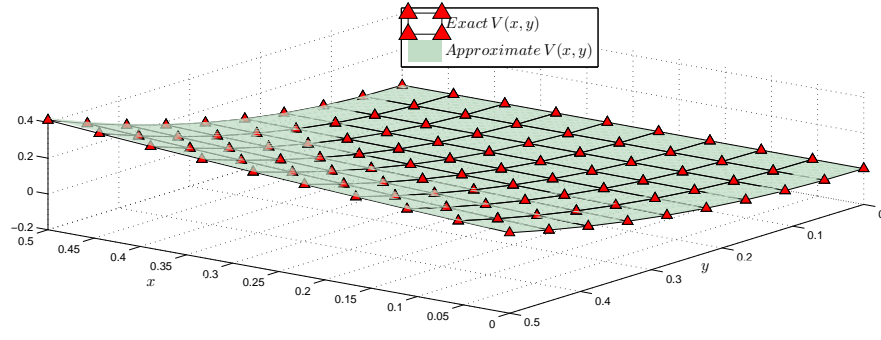


Figure 11: Comparing approximate solution  $V(x, y)$  with the exact solution of the system obtained using  $\alpha = 4, \beta = 3$  and  $M = 8$ .

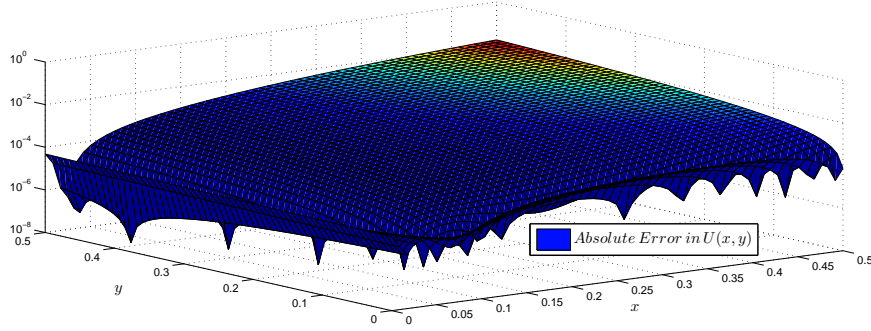


Figure 12: Error of approximation in  $U(x, y)$  with the component  $U(x, y)$  by keeping  $\alpha = 4, \beta = 3$  and  $M = 10$ .

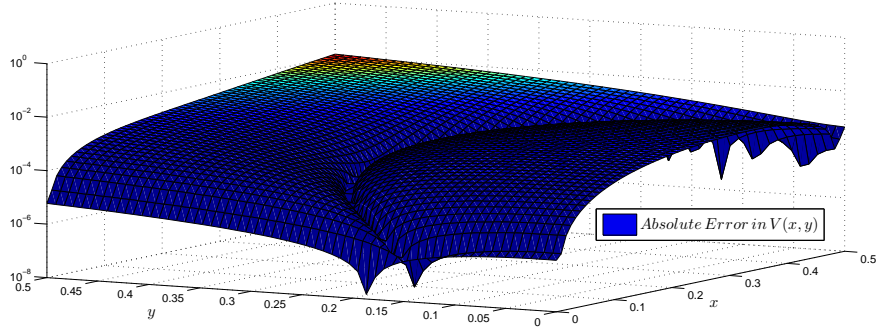


Figure 13: Error of approximation in  $U(x, y)$  with the component  $U(x, y)$  by keeping  $\alpha = 4, \beta = 3$  and  $M = 10$ .

## 6 Conclusion

Our proposed research work is an improved extension of the previously developed methods based on operational matrices, see for example, [8, 9, 18] and references therein. In [8, 9],

Legendre polynomials are used to develop the numerical scheme. We extend the method to more generalized two dimensional Jacobi orthogonal polynomials through the operational matrix of which we approximate the solution using two parameters.

The fundamental objective achieved through our method is to be able to tackle efficiently a generalized class of coupled system of FOPDEs having mixed partial derivative terms, without actually having to discretize the problem. The objective has been achieved by developing a method based on the orthogonal shifted JPs in combination with the operational matrices of Riemann-Liouville fractional integral and Caputo fractional derivatives for shifted JPs. It has been observed that the results are in a good agreement with the exact solution with a low number of approximating terms.

It is also noted that the numerical solutions approach the solutions for problems as the order of the fractional derivative approaches to 1, for a fixed  $\sigma_1$  and  $\sigma_2$ . The proposed method has the advantage of transforming the problem into the system of algebraic equations which greatly simplifies the task of finding the solution of FOPDEs. This theory can also be applied to solve the problems numerically existing in fluid dynamics also this theory can be applied to many other linear and nonlinear problems of fractional order. For the simulations, we use the software Matlab. Readers are welcome to communicate to us their feedback and comments.

## 7 Competing interests

The authors declare that they have no competing interests.

## 8 Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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